Geometric Realizations of Hecke algebras & an overview (Nikolay 1/15/25)

§1 Introduction.

(May work w/ F = nonarchimedean local field, $\mathcal{O}$ = ring of integers)

Let $\mathbb{Q}_p$ = p-adic field
$\mathbb{Z}_p \subseteq$ = ring of intergers
$k = \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$ residue field
G = split reductive linear algebraic group over k (closed subgp of GL_N).
$B \subseteq$ = Borel subgroup
Let I = Iwahori = preimage of $B \leq G$ under $ev: G(\mathbb{Z}_p) \to G$

Def (Iwahori-Hecke algebra of $G(\mathbb{Q}_p)$)

Let $\mathbb{C}[I \backslash G(\mathbb{Q}_p)/I] = \left\{ f: G(\mathbb{Q}_p) \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ compactly supported and} \\ I\text{-bi-equivariant} \end{array} \right\}$

Equip w/ algebra structure coming from convolution product:

$$f_1 * f_2 = \int_{G(\mathbb{Q}_p)} f_1(y) f_2(y^{-1}x)\, dy$$

locally compact topological group

- Bruhat decomposition $\Rightarrow$ $\mathbb{C}[I \backslash G(\mathbb{Q}_p)/I]$ has basis indexed by elements of $W^{aff} = W \ltimes \underbrace{Q^\vee}_{\text{coroot lattice of } G}$ ✓

(Assume G^{sc} so W^{aff} is Coxeter gp)

Def (Bernstein) Let (W,S) = Coxeter group. The Hecke algebra H associated to (W,S) has a free $\mathbb{Z}[q^{\pm 1}]$-basis given by $\{T_w : w \in W\}$ such that

(1) $(T_s+1)(T_s-q)=0$ if $s \in S$ is simple reflection

(2) $T_y \cdot T_w = T_{yw}$ if $\ell(y)+\ell(w) = \ell(yw)$.

Thus, $\mathbb{H}(W) = q$-deformation of $\mathbb{Z}[W]$.

($\mathbb{H}$ denotes affine Hecke algebra coming from W^{aff}).

If $W = W^{\text{aff}}$, then have $\mathbb{H}$ has basis $\{X^\lambda \cdot T_w : \lambda \in Q^\vee, w \in W^{\text{fin}}\}$

& (1) $\mathbb{Z}[q^{\pm1}]\langle T_w \rangle \simeq H_W^{\text{fin}}$

(2) $\mathbb{Z}[q^{\pm1}]\langle X^\lambda \rangle \simeq \mathbb{Z}[q^{\pm1}][x_1^{\pm1}, \dots, x_n^{\pm1}]$

(3) $T_s X^\lambda = X^\lambda T_s$ if $\langle \lambda, \alpha_s^\vee \rangle = 0$

(4) $T_s X^{s(\lambda)} T_s = q X^\lambda$ if $\langle \lambda, \alpha_s^\vee \rangle = 1$

Fact $\mathbb{C}[I \backslash G(\mathbb{Q}_p)/I] \simeq \mathbb{C} \otimes_{\mathbb{Z}[q^{\pm1}]} \mathbb{H}(\check{W}^{\text{aff}})$. (1st appearance of duality)

$q^{?} \delta_{IwI} \longleftarrow T_w$

The primary goal of Deligne-Lusztig theory is to classify irreps of $H(\check{W}^{\text{aff}})$. This is related to classification of infinite-dim irreps of $G(\mathbb{Q}_p)$, for which we have:

Deligne-Langlands Conjecture (LLC for finite fields)

$$\left\{\text{Simple } \mathbb{H}(\check{W}^{\text{aff}})\text{-modules}\right\} \simeq \left\{\begin{matrix}\text{Irreducible } G(\mathbb{Q}_p)\text{-mod} \\ \text{w/ } I\text{-fixed vectors}\end{matrix}\right\} \longleftarrow\text{-}\text{-}\longrightarrow \left\{\begin{matrix}\text{certain "tame" reps} \\ \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to G^\vee\end{matrix}\right\}$$

$$V^I \longleftarrow V$$

The classification of irred H-mod was accomplished by Kazhdan-Lusztig and Ginzburg, & 1st step is to realize H geometrically:

§2 K-theory of Steinberg

Let X be a quasi-projective G-variety.

Let $\text{Coh}^G(X)$ = category of G-equivariant coherent sheaves on X

$K^G(X) :=$ Grothendieck group of $\text{Coh}^G(X)$.

$= \mathbb{Z}[\text{isom}(\text{Coh}^G(X))] / ([B] = [A] + [C])$

for any $0 \to A \to B \to C \to 0$ SES.

Rep. ring of G.

(19 min)

Ex (1) $K^G(pt) = K^0(\mathrm{Rep}\, G) =: R(G)$, $\mathbb{C} \otimes_{\mathbb{Z}} K^0(\mathrm{Rep}\, G) \simeq \mathcal{O}(G)^G$ (G reductive)

class function on G.

(2) $K(\mathbb{C}^*) = \mathbb{Z}$, $K^{\mathbb{C}^*}(\mathbb{C}^*) = \mathbb{Z}[q^{\pm 1}]$

(3) $K(\mathrm{Coh}^G(G/H)) = K(\mathrm{Coh}^H(pt)) = \mathbb{Z}[\mathrm{Irr}(H)] = R(H)$.

H closed subgp

(4) Let $\tilde{\mathcal{N}} = T^*(G/B)$ = Springer resolution.

Representation ring of T.

$\Rightarrow K^G(\tilde{\mathcal{N}}) = K^G(G \times^B \mathrm{Lie}(n)) \overset{K(\text{vector space}) = K(pt)}{=} K^B(pt) = R(T)$.

Now, for G-varieties X of the form $X = Z \times_Y Z$, ($Z \to Y$ proper) there exists natural

convolution product $K^G(X) \otimes K^G(X) \longrightarrow K^G(X)$.

- Need p_{13} proper to preserve coherence
- p_{12}^*, p_{23}^* : Need locally complete intersection

given by $\boxed{\mathcal{F} * \mathcal{G} := (p_{13})_*(p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{G}).}$

The key property is from set theory:

$$\mathrm{Im}(p_{13} : p_{12}^{-1}(Z \times_Y Z) \cap p_{23}^{-1}(Z \times_Y Z) \longrightarrow Z \times Z) = Z \times_Y Z$$

where $p_{ij} : Z \times Z \times Z \longrightarrow Z \times Z$

Also, for <u>smooth</u> X, have $\otimes$-product: (Need $\otimes$ to be exact functor on coherent sheaves)

$$\mathcal{F} \otimes \mathcal{G} := \Delta^*(\mathcal{F} \boxtimes \mathcal{G}) \quad \text{for } \Delta : X \longrightarrow X \times X.$$

Let $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \cong \{(x, b, b') \in \mathfrak{g} \times \mathcal{B} \times \mathcal{B} \mid x \in b \cap b'\}$ = Steinberg variety

$\tilde{\mathcal{N}} \times^{L}_{\mathfrak{g}} \tilde{\mathcal{N}}$ correct for derived. (For Bez.)

<u>Main Thm</u> There is algebra isomorphism $K^{G \times \mathbb{C}^*}(Z) \simeq H$

- By specializing $q = 1$, we get $K^G(Z) \simeq \mathbb{Z}[W^{\mathrm{aff}}]$

• By specializing $q=p$ we get $K^{G^\vee\times\mathbb{C}^*}(Z)\big|_{q=p} \simeq \mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$

<u>§ SL_2</u> We may verify Main Thm via directly defining map on generators & showing it factors through all relations.

$$\mathbb{H} = \mathbb{Z}[q^{\pm1}]\langle X^{\pm1}, T\rangle \Big/ \left(\begin{matrix}(T+1)(T-q)=0, & XX^{-1}=X^{-1}X=1 \\ TX^{-1}-XT=(1-q)X & \end{matrix}\right)$$

Steinberg $Z = \underbrace{T^*_{\mathbb{P}_\Delta}(\mathbb{P}\times\mathbb{P})}_{= Z_\Delta} \coprod \underbrace{T^*_{\overline{\mathbb{P}\times\mathbb{P}\setminus\mathbb{P}_\Delta}}(\mathbb{P}\times\mathbb{P})}_{= Z_Y}$ (union of conormals to $T^*\mathcal{B}\times\mathcal{B}$ indexed by G_Δ-orbits)

Z_Y = zero section of $T^*(\mathbb{P}\times\mathbb{P})$, $Y=\mathbb{P}\times\mathbb{P}\setminus\mathbb{P}_\Delta$.

Let $Q := \pi_Y^*\,\Omega^1_{\mathbb{P}\times\mathbb{P}/\mathbb{P}}$, $\quad \pi_Y : Z_Y \longrightarrow \mathbb{P}\times\mathbb{P}$ (is isom)

$\mathcal{O}_n = \pi_\Delta^*\,\mathcal{O}(n)$ $\quad \pi_\Delta : Z_\Delta \longrightarrow \mathbb{P}_\Delta$

Define $\theta : \{X^{\pm1}, T\} \longrightarrow K^{G\times\mathbb{C}^*}(Z)$

$$X \longmapsto [\mathcal{O}_{-1}]$$
$$X^{-1} \longmapsto [\mathcal{O}_1]$$
$$T \longmapsto -[qQ] - [\mathcal{O}_0]$$

<u>Lemma</u> θ satisfies all $\mathbb{H}$ relations:

(1) $[qQ]*[qQ] = -(q+1)q\,Q$

(2) $[qQ]*\mathcal{O}_1 - \mathcal{O}_{-1}*(qQ) = q\,\mathcal{O}_{-1} - \mathcal{O}_1$

(3) $\mathcal{O}_1 * \mathcal{O}_{-1} = \mathcal{O}_0$.

<u>Pf</u> $Q \cong \Omega_{\mathbb{P}\times\mathbb{P}/\mathbb{P}} \simeq \mathcal{O}_{\mathbb{P}} \boxtimes \Omega^1_{\mathbb{P}}$.

Have Koszul complex: $\mathbb{P} \underset{\pi}{\overset{i}{\rightleftarrows}} T^*\mathbb{P}^1$

restrict $\omega\in\Omega^1$ to 0 section

$$0 \to \mathcal{O}_{T^*\mathbb{P}^1} \xrightarrow{\delta} \pi^*\Omega^1_{\mathbb{P}} \longrightarrow i_*\Omega^1_{\mathbb{P}} \longrightarrow 0$$

$$\mathcal{O}_{\mathbb{P}}\boxtimes\big(\quad 0 \longrightarrow (\mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{T^*\mathbb{P}^1})q^{-1} \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}} \boxtimes \pi^*\Omega^1_{\mathbb{P}} \longrightarrow Q \longrightarrow 0 \qquad \mathbb{P}^1\times T^*\mathbb{P}^1$$

To make $\mathbb{C}^*$-equivariant, get following:

$$_q Q = q\cdot(\mathcal{O}_{\mathbb{P}} \boxtimes \pi^*\Omega^1_{\mathbb{P}}) - \mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{T^*\mathbb{P}} \in K^{G\times\mathbb{C}^*}(\mathbb{P}\times T^*\mathbb{P})$$

This allows to verify (1) & (2) ∎

Heuristic comment: ad hoc tricks to compute convolution product

Now, have alg hom $\Theta : \mathbb{H} \longrightarrow K^{G\times\mathbb{C}^*}(Z)$.

By construction, $\mathbb{H}_0 := \langle X^{\pm 1}\rangle \underset{\sim}{\overset{\Theta}{\longrightarrow}} K^{G\times\mathbb{C}^*}(Z_\Delta)$.

(Cellular fibration) Have SES

$$0 \longrightarrow \underbrace{K^{G\times\mathbb{C}^*}(Z_\Delta)}_{\substack{\| \\ \mathbb{H}_0}} \longrightarrow K^{G\times\mathbb{C}^*}(Z) \longrightarrow \underbrace{K^{G\times\mathbb{C}^*}(\overbrace{T^*_Y(\mathbb{P}\times\mathbb{P})}^{Z\setminus Z_\Delta})}_{\substack{\| \\ K^{G\times\mathbb{C}^*}(Y) = K^{G\times\mathbb{C}^*}(\mathbb{P}) \\ = R(T\times\mathbb{C}^*)}} \longrightarrow 0$$

Thus, $\Theta : \mathbb{H}/\mathbb{H}_0 \xrightarrow{gr\,\Theta} K^{G\times\mathbb{C}^*}(Z)/K^{G\times\mathbb{C}^*}(Z_\Delta) \xrightarrow{\sim} K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P}))$.

$\Longrightarrow gr\,\Theta$ isom $\Longrightarrow \Theta$ isom ∎

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§ General G Proof strategy.

① Construct a map $\Theta : \mathbb{H} \longrightarrow K^{G\times\mathbb{C}^*}(St)$ on generators & check relations hold by finding a faithful (polynomial) rep. of both algebras & seeing each a & $\Theta(a)$, $a\in\mathbb{H}$, act the same way.

Indeed, define

$$\Theta : \mathbb{H} \longrightarrow K^{G\times\mathbb{C}^*}(Z)$$

$s\in$ Simple reflects $T_s \longmapsto -\left(q\cdot[\pi_s^*\Omega_{Y_s/\mathcal{B}}] + \mathcal{O}_\Delta\right)$

$\lambda\in$ cocharacter lattice $X^\lambda \longmapsto \Delta_*(p^*\mathcal{O}_\lambda)$

$\left\{ \begin{array}{l} Y_s \in \mathcal{B}\times\mathcal{B} \text{ Bruhat cell corresponding to } s. \\ \pi_s : T^*_{Y_s}(\mathcal{B}\times\mathcal{B}) \longrightarrow \bar{Y}_s \end{array} \right.$ smooth irred. comp. of Z.

$\left\{ \begin{array}{l} \tilde{\mathcal{N}} \xrightarrow{\Delta} \tilde{\mathcal{N}}\times_{\mathcal{N}}\tilde{\mathcal{N}} = Z \\ p\downarrow \\ G/\mathcal{B} \end{array} \right.$

Let $e = \sum_{w \in W^{fin}} T_w \in \mathbb{H}$, $\quad \mathbb{H}e \overset{⊛}{=} \mathrm{Ind}_{H_W}^{\mathbb{H}} \varepsilon$ as left $\mathbb{H}$-mod

where $\varepsilon : H_W \longrightarrow \mathbb{Z}[q^{\pm 1}]$, $\quad T_w \longmapsto q^{\ell(w)}$

$$\rho : \mathbb{H} \longrightarrow \mathrm{End}_{\mathbb{Z}[q^{\pm1}]}(\mathbb{H}e)$$

OTOH, $K^{G\times\mathbb{C}^*}(T^*\mathcal{B}) \overset{\text{(from before)}}{=\!=} \overset{R(T)[q^{\pm1}]}{\mathbb{Z}[q^{\pm1}][T]} \overset{⊛}{\cong} \mathbb{H}e$

$K^{G\times\mathbb{C}^*}(Z)$ ↻ convolution

Thus:

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\theta"] \arrow[d] & K^{G\times\mathbb{C}^*}(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}) \arrow[d] \\ \mathrm{End}_{\mathbb{Z}[q^{\pm1}]} \mathbb{H}e \arrow[r, "\sim"', "\text{isom of alg}"] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{End}_{\mathbb{Z}[q^{\pm1}]}(T^*\mathcal{B}) \end{tikzcd}$$

$\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = St$

$\Rightarrow \theta$ is alg hom

(2) Introduce Filtrations & show $gr(\theta)$ isom: For $w \in W^{fin}$,

$$\mathbb{H}_{\leq w} := \mathrm{span}\{ X^\lambda T_y : \lambda \in P,\ y \leq w \}$$

Then $\mathbb{H}_{\leq w} / \mathbb{H}_{< w} \cong \mathbb{Z}[q^{\pm1}][T] \cdot T_w$ as modules.

And,

$$Z_{\leq w} = \coprod_{y \leq w} T^*_{Y_w}(\mathcal{B}\times\mathcal{B})$$ is $G\times\mathbb{C}^*$-stable!

Then $$K^{G\times\mathbb{C}^*}(Z_{\leq w}) / K^{G\times\mathbb{C}^*}(Z_{<w}) = K^{G\times\mathbb{C}^*}(T^*_{Y_w}(\mathcal{B}\times\mathcal{B}))$$
$$= \mathbb{Z}[q^{\pm1}][T] \cdot \underbrace{[\mathcal{O}_{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}]}_{\text{generator}}$$

Then $\theta : \mathbb{H} \to K^{G\times\mathbb{C}^*}(Z)$ is filtration preserving and

$$gr_w \theta : \mathbb{H}_{\leq w} / \mathbb{H}_{<w} \xrightarrow{\sim} K(Z_{\leq w}) / K(Z_{<w})$$
$$T_w \longmapsto \underbrace{c_w}_{\neq 0} \cdot [\mathcal{O}_{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}]$$

$\Longrightarrow \theta$ is isomorphism ■ BREAK!

§ Geometric Local Langlands Conjecture

Recall main result of [CG] is

$$(0) \qquad \mathbb{C}[I\backslash G(\mathbb{Q}_p)/I] \simeq K^{\check{G}\times\mathbb{C}^*}(\check{Z})\Big|_{q=p}$$ 0-categorical

Grothendieck's Function/sheaf dichotomy says we may replace functions on $I\backslash G(\mathbb{Q}_p)/I$ with (mixed ℓ-adic perverse) sheaves on affine flag variety, $Fl := G(\overline{\mathbb{F}}_p((t)))/I$. [CG] then conjecture a categorical equivalence $Shv(Fl) \simeq Coh^{\check{G}}(\check{Z})$. This was proved by Bezrukavnikov:

Thm (Bezrukavnikov). Let $\check{Z} = \tilde{\mathcal{N}} \overset{L}{\underset{\check{g}}{\times}} \tilde{\mathcal{N}}$ be the Steinberg derived stack. Then there exists an equivalence of categories

$$(1) \qquad \underbrace{Shv^{I}(Fl)}_{\substack{I\text{-equivariant derived category}\\ \text{of } \ell\text{-adic sheaves on } Fl\\ \text{(with bounded \& fin. dim cohom)}}} \simeq D^b(Coh^{\check{G}}(\check{Z}))$$ (1-categorical)

Taking K-theory of (1) recovers the isom of affine Hecke algebras in (0), which holds for all q! (Do not need to specialize $q=p$.)

The goal of Geometric Local Langlands (GLLC) is to replace $\overline{\mathbb{F}}_p((t))$ with $\mathbb{C}((t))$ (de-Rham). Then, the conjectural equivalence is of 2-categories:

Conjecture (Gaitsgory) (GLLC.) There exists a functor

$$(2) \quad (D\text{-mod}(LG))\text{-mod} \xrightarrow{\mathbb{L}} \mathrm{ShvCat}(\mathrm{LocSys}_{\check{G}}(\mathring{D}))$$

$\mathbb{L}$ ← Langlands functor

(2-categorical)

which induces an equivalence of categories upon appropriate modifications(*)

Let's understand both sides & show applications of the conjectural equivalence (2):

§ Categorical Representations (Informal introduction)

Def A k-linear dg-category $\mathcal{C}$ is a collection of objects such that

(i) For each $c_1, c_2 \in \mathrm{Ob}(\mathcal{C})$, have a complex of k-vector spaces ($\mathbb{Z}$-graded) $\mathrm{Hom}(c_1, c_2)$

(ii) For each $c_1, c_2, c_3 \in \mathrm{Ob}(\mathcal{C})$, have composition

$$\mathrm{Hom}(c_1, c_2) \otimes \mathrm{Hom}(c_2, c_3) \longrightarrow \mathrm{Hom}(c_1, c_3)$$

which is associative

A dg-cat is cocomplete if closed under shifts, cones, & arbitrary $\oplus$.

$$\underset{((\infty,1)\text{-cat})}{\mathrm{DGCat}} = \begin{cases} \text{objects} = \text{cocomplete dg-cats} \\ \text{1-morp} = \text{continuous (commute w/ colimit) functors.} \end{cases}$$

Key fact: DGCat is unital symmetric monoidal (with respect to Lurie $\otimes$)

Ex QCoh(X), D-mod(X), & Vect $\in$ DGCat, and

$$\mathrm{QCoh}(X)\otimes \mathrm{QCoh}(Y) = \mathrm{QCoh}(X\times Y) \qquad \mathcal{M}\boxtimes\mathcal{N} \simeq \pi_X^*\mathcal{M}\otimes\pi_Y^*\mathcal{N}$$
$$\text{D-mod}(X)\otimes\text{D-mod}(Y) = \text{D-mod}(X\times Y) \qquad \mathcal{M}\boxtimes\mathcal{N} = \pi_X^!\mathcal{M}\overset{!}{\otimes}\pi_Y^!\mathcal{N}$$

$$\underbrace{\mathrm{Vect}}_{=\text{D-mod}(pt)}\otimes\,\mathcal{C} = \mathcal{C}, \quad \forall\, \mathcal{C}\in \mathrm{DGCat}$$

Def An algebra object of DGCat is a monoidal dg cat, i.e., $\mathcal{A}\in$ DGCat equipped with functor $\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ which is associative (up to coherent homotopy)

Ex For algebraic variety Y, $\mu: Y\times Y\to Y$, QCoh(Y) & D-mod(Y) both acquire "convolution" monoidal structure via μ_*.

Def Given monoidal dg-cat $\mathcal{A}$, a module $\mathcal{M}\in\mathcal{A}$-mod is $\mathcal{M}\in$ DGCat such that

$$\mathcal{A}\otimes\mathcal{M}\to\mathcal{M}$$

is "associative" up to coherent homotopy.
(satisfies module axioms)

Ex Y is G-variety. Then $act: G\times Y\to Y$ induces

$$act_*: \text{D-mod}(G\times Y)\to\text{D-mod}(Y) \Rightarrow \text{D-mod}(Y)\in(\underset{\text{algebra}}{\text{D-mod}(G)})\text{-mod}$$
$$act_*: \mathrm{QCoh}(G\times Y)\to\mathrm{QCoh}(Y) \Rightarrow \mathrm{QCoh}(Y)\in(\mathrm{QCoh}(G))\text{-mod}$$

Warning: Coh(Y) is not algebra

Def Let LG = loop group. So, for any $\mathbb{C}$-algebra R,

$$LG(R) = G(R((t))). \qquad \Big(\text{So, } LG = \underset{\text{mapping stack.}}{\mathrm{Maps}}(\mathcal{D}^\times, G)\Big)$$

Then, $\text{D-mod}(LG) := \lim \text{D-mod}(K \backslash LG / K)$ $\Rightarrow$ Resembles compactly supported distributions on $G(\mathbb{F}_q((t)))$, (i.e, the Hecke algebra!)

$K < G$ compact open subgps

As in finite-dim. case, $\text{D-mod}(LG)$ is monoidal DG Cat

$\Longrightarrow$ $(\text{D-mod}(LG))$-mod makes sense!

Finally, we define sheaves of categories:

Def Let $X \overset{=\ \mathrm{LocSys}_{G^\vee}(D^\times)}{}$ be a space. Then $\mathrm{Shv}(\mathrm{Cat}(X))$ is the association

for any $f : \overset{\text{affine}}{S} \longrightarrow X$, a module $M \in \underbrace{\mathrm{QCoh}(S)}_{\text{monoidal cat}}$-mod

So If X is affine, $\mathrm{Shv}(\mathrm{Cat}(X)) = \mathrm{QCoh}(X)$-mod

If X is pt, $\mathrm{Shv}(\mathrm{Cat}(pt)) = \mathrm{DG\ Cat}.$

Now, we return to GLLC & discuss applications:

$$L : (\text{D-mod}(LG))\text{-mod} \longrightarrow \mathrm{Shv\ Cat}(\mathrm{LocSys}_{G^\vee}(D^\times))$$

$$\mathcal{C} \longmapsto \mathcal{C}^{(LN, \psi)}$$

$$\parallel$$

$$\mathrm{Hom}$$

Conjecturally:

$$\text{D-mod}(\mathrm{Gr}_G) \longmapsto \mathrm{QCoh}(pt/G^\vee) = \mathrm{Rep}(G^\vee)$$

$\mathrm{Gr}_G = LG/L^+G$

$$\text{D-mod}(\mathrm{Fl}_G) \longmapsto \mathrm{QCoh}(\tilde{N}/G^\vee)$$

$\mathrm{Fl}_G = LG/I$

$$\text{D-mod}(LG/LN, \psi) \longmapsto \mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}(D^\times))$$

$$\text{D-mod}(\mathrm{Bun}_G^{\infty x}) \longmapsto \mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}(X \setminus x))$$

Now, equivalence of cat $\Rightarrow$ $\forall\, \mathcal{C}, \mathcal{D}$, $\mathrm{Hom}_{D\text{-mod-}}(\mathcal{C}, \mathcal{D}) = \mathrm{Hom}_{\mathrm{ShvCat-}}(\mathbb{L}(\mathcal{C}), \mathbb{L}(\mathcal{D}))$

Fact $\mathrm{Hom}_{\mathrm{QCoh}(Z)}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) = \mathrm{QCoh}(X \times_Z Y)$ (⊛)

(⊛) & Similarly for D-modules.

Applications

① [B, CG]:

$$\mathrm{End}_{LG}(D\text{-mod}(LG/I)) \overset{⊛}{=} D\text{-mod}(I \backslash LG / I) \qquad LG/I \times_{LG} LG/I = I^{\backslash LG/I}$$

$$\| \; \mathbb{L}$$

$$\mathrm{End}(\mathrm{QCoh}(\tilde{N}/G^\vee)) \overset{⊛}{=} \mathrm{IndCoh}_{nilp}(\tilde{N}/G^\vee \underset{\check{g}/G^\vee}{\times} \tilde{N}/G^\vee) = \mathrm{IndCoh}^{G^\vee}_{nilp}(\check{Z})$$

$\check{g}/G^\vee$ Replaces $\mathrm{LocSys}_{G^\vee}(D^\times)$

② Derived Geometric Satake

$$\mathrm{End}(D\text{-mod}(\mathrm{Gr}_G)) = D\text{-mod}(L^+G \backslash LG / L^+G)$$

$$\| \; \mathbb{L}$$

$$\mathrm{End}(\mathrm{QCoh}(pt/G^\vee)) = \mathrm{IndCoh}^{G^\vee}_{nilp}(pt \times_{g^\vee} pt)$$

③ [ABG]

$$\mathrm{Hom}(D\text{-mod}(\mathrm{Gr}_G), D\text{-mod}(\mathrm{Fl}_G)) = D\text{-mod}(L^+G \backslash LG / I)$$

$$\| \; \mathbb{L}$$

$$\mathrm{QCoh}(pt/G^\vee \underset{\check{g}/G^\vee}{\times} \check{n}/\check{B}) = \mathrm{QCoh}^{G^\vee}(pt \times_{g^\vee} \tilde{N}^\vee)$$

④ [AR]

$$\mathrm{Hom}(D\text{-mod}(LG/LN, \psi), D\text{-mod}(LG/I)) = D\text{-mod}(I \backslash LG/N, \psi)$$

$$\| \; \mathbb{L}$$

$$\mathrm{Hom}(\mathrm{QCoh}(LS(D^\times)), \mathrm{QCoh}(\check{N}/G^\vee)) = \mathrm{QCoh}(N^\vee/G^\vee).$$

(5) (Unramified GLC)

$$\mathrm{Hom}_{LG}\left(D(Gr_G), D(Bun_G^{\infty x}) \right) \overset{\text{Uniformization}}{=} D\text{-mod}(Bun_G)$$

$$\| \; \mathbb{L}$$

$$QCoh\left(LS_{G^\vee}(D) \underset{LS_{G^\vee}(D^\circ)}{\times} LS_{G^\vee}(X \setminus x) \right) = QCoh(LS_{G^\vee}(X))$$